Finite Element Model Updating Using Primal-Relaxed Dual Global Optimization Algorithm

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ABSTRACT

Finite element (FE) modeling has become a powerful tool in predicting the response of various engineering structures. However, predictions from the numerical model often differ from in-situ experimental measurements due to numerous approximations and inaccuracies in the model. The in-situ experimental data obtained from the as-built structure can be used to update selected model parameters to obtain a more accurate FE model that truly reflects the behavior of the as-built structure. This research investigates FE model updating by the modal property difference approach using eigenvalues and eigenvectors. The modal property difference approach is a nonconvex optimization problem, for which generic solvers cannot guarantee global optimality. However, the problem can be reformulated into a biconvex problem so that the global optimum can be found using a primal-relaxed dual (P-RD) decomposition approach. The formulation of the model updating algorithm and examples that demonstrate its functionality are presented in this paper.

INTRODUCTION

FE model updating refers to identifying the actual values of model parameters to better capture the behavior of an as-built structure. This naturally lends itself to a mathematical optimization problem where model parameters are optimized to minimize the difference between the analytically predicted behaviors and experimental vibration. To date, the most commonly studied optimization objectives are based in the frequency-domain, i.e. attempting to minimize the difference between the analytical and experimental modal properties. In general, the problem is nonconvex and global optimality cannot be guaranteed using typical gradient search algorithms. To increase the likelihood of finding the global optimum, previous researchers used gradient descent methods with multiple, randomly generated starting points [1]. Stochastic global optimization algorithms, such as the genetic algorithm and simulated annealing found

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in MATLAB's Global Optimization Toolbox, are less likely to be trapped in local optima but cannot guarantee that a global optimum has been found. In contrast, deterministic global optimization algorithms take advantage of the mathematical structure of the problem and guarantee global optimality of the solution or provides a certificate of the tolerance in case of nonconvergence due to a limitation in computing resources. For the modal property difference approach in FE model updating [1], its biconvexity can be exploited and global optimality can be guaranteed using the primal-relaxed dual (P-RD) decomposition approach [2]. The problem reformulation is first presented, followed by a summary of the P-RD algorithm. Finally, two examples are provided that show the validity of the model updating algorithm.

MODEL UPDATING FORMULATION

To perform model updating, the stiffness matrix for a structure with N-degrees of freedom (DOFs) is first denoted by $\mathbf{K}(\boldsymbol{\alpha}) = \mathbf{K}_0 + \sum_{j=1}^{n_{\boldsymbol{\alpha}}} \alpha_j \mathbf{K}_j$ where $\mathbf{K}_0 \in \mathbb{R}^{N\times N}$ is the nominal stiffness matrix; α_j is the j-th entry of the stiffness updating variable $\boldsymbol{\alpha} \in \mathbb{R}^{n_\alpha}$, which represents the relative change of a stiffness parameter from the nominal value; and $\mathbf{K}_j \in \mathbb{R}^{N\times N}$ is the influence matrix corresponding to α_j. It is assumed that the mass matrix, $\mathbf{M} \in \mathbb{R}^{N\times N}$, is accurate enough and does not require updating.

The formulation considered in this research is based on the generalized eigenvalue problem in structural dynamics:

$$[\mathbf{K}(\boldsymbol{\alpha}) - \lambda_i \mathbf{M}]\{\boldsymbol{\psi}_i\} = \mathbf{0}, \quad i = 1, \dots, n_{\text{modes}} \tag{1}$$

where $\lambda_i \in \mathbb{R}$ and $\boldsymbol{\psi}_i \in \mathbb{R}^N$ are the i-th eigenvalue and eigenvector, respectively, and n_{modes} denotes the number of modes. We can write $\lambda_i(\boldsymbol{\alpha})$ and $\boldsymbol{\psi}_i(\boldsymbol{\alpha})$ since λ_i and $\boldsymbol{\psi}_i$ implicitly depend on $\boldsymbol{\alpha}$. In general, eigenvectors extracted from experimental data are limited to DOFs measured by sensors. To account for this, we define $\boldsymbol{\psi}_i^{\mathcal{M}}(\boldsymbol{\alpha}) \in \mathbb{R}^{n_{\mathcal{M}}}$ that represents entries of $\boldsymbol{\psi}_i(\boldsymbol{\alpha})$ at $n_{\mathcal{M}}$ measured DOFs. The experimentally obtained eigenvalues and eigenvectors are denoted as $\lambda_i^{\text{EXP}} \in \mathbb{R}$ and $\boldsymbol{\psi}_i^{\text{EXP},\mathcal{M}} \in \mathbb{R}^{n_{\mathcal{M}}}$, respectively. In this research, the experimental eigenvector, $\boldsymbol{\psi}_i^{\text{EXP},\mathcal{M}}$, is normalized so that the entry with the largest absolute value equals one. Accordingly, the same entry of $\boldsymbol{\psi}_i^{\mathcal{M}}(\boldsymbol{\alpha})$ is normalized to be one.

The modal property difference formulation with the eigenvector difference approach attempts to directly minimize the difference between the experimental and analytical eigenvalues and eigenvectors [3]:

$$\begin{aligned} \underset{\boldsymbol{\alpha}}{\text{minimize}} \quad & \sum_{i=1}^{n_{\text{modes}}} \left\{ \left(\frac{\lambda_i^{\text{EXP}} - \lambda_i(\boldsymbol{\alpha})}{\lambda_i^{\text{EXP}}} \cdot w_{\lambda_i} \right)^2 + \left\| \{\boldsymbol{\psi}_i^{\text{EXP},\mathcal{M}} - \boldsymbol{\psi}_i^{\mathcal{M}}(\boldsymbol{\alpha})\} \cdot w_{\boldsymbol{\psi}_i} \right\|_2^2 \right\} \\ \text{subject to} \quad & \mathbf{L}_{\boldsymbol{\alpha}} \le \boldsymbol{\alpha} \le \mathbf{U}_{\boldsymbol{\alpha}} \end{aligned} \tag{2}$$

where w_{λ_i} and $w_{\boldsymbol{\psi}_i}$ are weighting factors and $\mathbf{L}_{\boldsymbol{\alpha}}$ and $\mathbf{U}_{\boldsymbol{\alpha}}$ are lower and upper bounds on $\boldsymbol{\alpha}$, respectively. Recall that the implicit functions $\lambda_i(\boldsymbol{\alpha})$ and $\boldsymbol{\psi}_i^{\mathcal{M}}(\boldsymbol{\alpha})$ are originated from the eigenvalue problem in Eq. (1). As a result, the objective function and the

optimization problem in Eq. (2) are nonconvex, for which local search algorithms cannot guarantee global optimality.

Since the objective function in Eq. (2) is implicit on $\boldsymbol{\alpha}$, the problem needs to be reformulated to apply the primal-relaxed dual (P-RD) algorithm, which requires explicit objective and constraint functions. The implicit constraints of the generalized eigenvalue equation in Eq. (1) can be converted to an explicit form by introducing the analytical eigenvalue variable $\boldsymbol{\lambda} = [\lambda_1, \cdots, \lambda_{n_{\text{modes}}}]^{\text{T}} \in \mathbb{R}^{n_{\text{modes}}}$; the analytical eigenvector variable $\boldsymbol{\psi} = \left[\boldsymbol{\psi}_1; \cdots; \boldsymbol{\psi}_{n_{\text{modes}}}\right] \in \mathbb{R}^{N \cdot n_{\text{modes}}}$ where a semicolon is used to vertically concatenate vectors; and the slack variable, $\delta \in \mathbb{R}$:

$$
\begin{aligned}
\underset{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{\psi},\delta}{\text{minimize}} \quad & \delta \\
\text{subject to} \quad & -\mathbf{1}\delta \leq \begin{bmatrix} \dfrac{\lambda_i^{\text{EXP}} - \lambda_i}{\lambda_i^{\text{EXP}}} \cdot w_{\lambda_i} \\ \left\{\boldsymbol{\psi}_i^{\text{EXP},\mathcal{M}} - \boldsymbol{\psi}_i^{\mathcal{M}}\right\} \cdot w_{\boldsymbol{\psi}_i} \\ w_{\text{eig}} \cdot [\mathbf{K}(\boldsymbol{\alpha}) - \lambda_i \mathbf{M}]\{\boldsymbol{\psi}_i\} \end{bmatrix} \leq \mathbf{1}\delta, \; i = 1, \ldots, n_{\text{modes}} \\
& \mathbf{L}_{\boldsymbol{\alpha}} \leq \boldsymbol{\alpha} \leq \mathbf{U}_{\boldsymbol{\alpha}} \\
& L_{\lambda_i} \leq \lambda_i \leq U_{\lambda_i}, \; i = 1, \ldots, n_{\text{modes}} \\
& \mathbf{L}_{\boldsymbol{\psi}_i} \leq \boldsymbol{\psi}_i \leq \mathbf{U}_{\boldsymbol{\psi}_i}, \; i = 1, \ldots, n_{\text{modes}}
\end{aligned}
\tag{3}
$$

where $\mathbf{1}$ denotes a vector of all ones with appropriate dimension; w_{eig} is a weighting factor for the generalized eigenvalue equation; L_{λ_i} and U_{λ_i} are lower and upper bounds on λ_i, respectively; and $\mathbf{L}_{\boldsymbol{\psi}_i}$ and $\mathbf{U}_{\boldsymbol{\psi}_i}$ are lower and upper bounds on $\boldsymbol{\psi}_i$, respectively. Since the generalized eigenvalue equation in Eq. (3) is not constrained to be exactly satisfied, the problem in Eq. (3) is an approximation of Eq. (2).

The only nonconvex term in Eq. (3) is the matrix-vector multiplication between $[\mathbf{K}(\boldsymbol{\alpha}) - \lambda_i \mathbf{M}]$ and $\{\boldsymbol{\psi}_i\}$. By grouping $\boldsymbol{\alpha}$, $\boldsymbol{\lambda}$, and δ as the so-called primal variable set and $\boldsymbol{\psi}$ as the relaxed dual variable, the optimization problem in Eq. (3) is found to be bilinear, which is a special case of biconvex. This means that when holding the primal variables constant, the optimization problem over the dual variable is a linear program (and thus convex), and vice versa for a constant dual variable. Therefore, the P-RD algorithm can be used to find a global optimum.

PRIMAL-RELAXED DUAL ALGORITHM

In general, a biconvex optimization problem has the form:

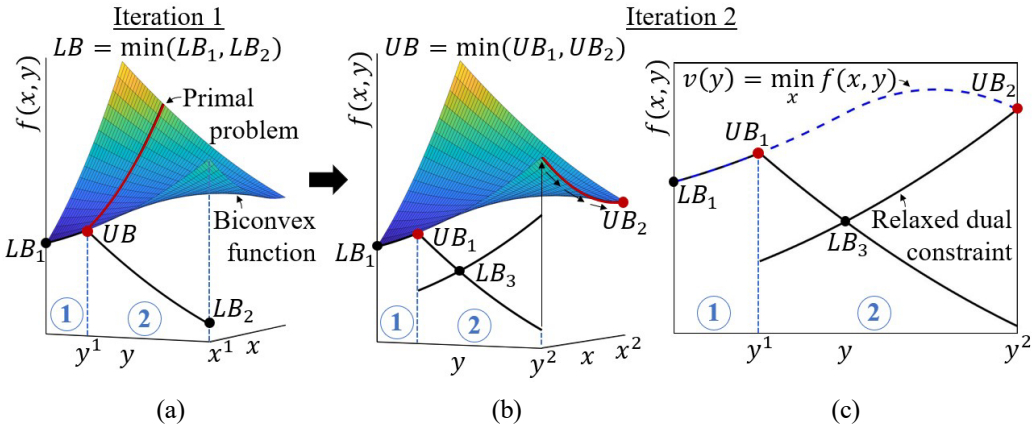
$$
\begin{aligned}
\underset{\mathbf{x},\mathbf{y}}{\text{minimize}} \quad & f(\mathbf{x}, \mathbf{y}) \\
\text{subject to} \quad & g_i(\mathbf{x}, \mathbf{y}) \leq 0, \; i = 1, \ldots, n_{\text{ineq}} \\
& h_i(\mathbf{x}, \mathbf{y}) = 0, \; i = 1, \ldots, n_{\text{eq}}
\end{aligned}
\tag{4}
$$

where for every fixed $\mathbf{y}$ value (and vice versa for every fixed $\mathbf{x}$), the functions $f(\mathbf{x}, \mathbf{y})$ and $g_i(\mathbf{x}, \mathbf{y})$ are convex in $\mathbf{x}$ and $h_i(\mathbf{x}, \mathbf{y})$ is affine in $\mathbf{x}$. For biconvex problems, the P-RD algorithm iteratively improves the upper and lower bound on the global optimum.

The algorithm converges when the upper bound from the primal problem, UB, is within a predefined tolerance, ξ, of the lower bound from the relaxed dual problem, LB. Constraints on the relaxed dual problem effectively partition the $\mathbf{y}$ variable space into subregions. Regions can be pruned if the lower bound in that region, LB_i, is greater than the current best upper bound. The region with the smallest lower bound is selected for further exploration in the next iteration.

In Figure 1(a), y is initially fixed to the value of y^1. This makes solving Eq. (4) much easier since it is convex, or linear in the case of Eq. (3). This problem is named the primal problem and the solution provides an upper bound on the global optimum. Using the Lagrange multipliers and the solution x^1 of the primal problem, a series of relaxed dual subproblems can be solved, corresponding to regions ① and ②. Since $LB_2 < LB_1$, region ② is considered for further examination. In Figure 1(b) and (c), the primal problem is solved with y^2 corresponding to LB_2. This provides a new upper bound, UB_2. Since y^2 is at a bound, only one subproblem needs to be solved and region ② is not subdivided. Due to the additional relaxed dual constraint, the lower bound increases from LB_2 in the first iteration to LB_3 in the second iteration. Figure 1(c) also shows the projection of the objective function onto the y variable space as $v(y)$.

In general, the primal problem is solved by fixing $\mathbf{y}$ in Eq. (4) to $\mathbf{y}^k$ at the k-th iteration. Each relaxed dual subproblem is defined over a particular partition of the $\mathbf{y}$ variable space and contains an accumulation of constraints up to the k-th iteration involving the Lagrangians of the primal problem. Since there is a collection of constraints from previous iterations, the relaxed dual problem provides tighter under-estimators each iteration. The value of $\mathbf{y}$ corresponding to the smallest regional lower bound, LB_i, is set as the new $\mathbf{y}^k$ in the primal problem. The primal and relaxed dual problems continue to be solved in an iterative fashion until the upper and lower bounds are within the predefined tolerance, ξ. The global optimum is guaranteed to be within the specified tolerance. Local searches may be performed in preprocessing and after each iteration to find a better upper bound. For a more detailed description of the algorithm and proof of convergence, see [2].



(a) (b) (c)

Figure 1. Illustration of the P-RD algorithm.

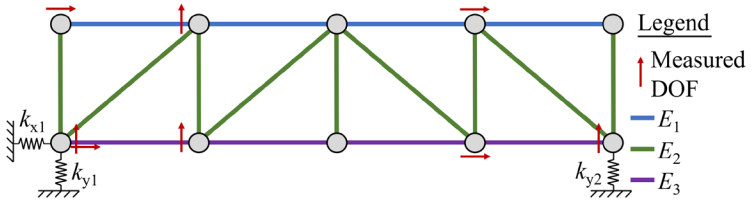


Figure 2. Plane truss structure.

TABLE I. STRUCTURAL PROPERTIES AND UPDATING RESULTS

Stiffness parameters		Nominal value	Actual value	Updating variable	α_j^{act}	α_j^*
Modulus of Elasticity ($\times 10^{11}$ N/m^2)	E_1	2.0	2.2	α_1	0.10000	0.10000
	E_2	2.0	1.8	α_2	-0.10000	-0.10000
	E_3	2.0	1.9	α_3	-0.05000	-0.05000
Spring Stiffness ($\times 10^6$ N/m^2)	k_{y1}	6.0	7.2	α_4	0.20000	0.20002
	k_{x1}	6.0	4.5	α_5	-0.25000	-0.25000
	k_{y2}	6.0	5.4	α_6	-0.10000	-0.10000

VALIDATION EXAMPLES

Numerical Study of Plane Truss Structure

The P-RD algorithm for the model updating formulation in Eq. (3) is first validated with a numerical simulation of the plane truss structure shown in Figure 2. The elastic moduli are divided into three groups: E_1 for the top bars, E_2 for the diagonal and vertical bars, and E_3 for the bottom bars. Flexible supports are modeled as springs, shown as k_{x1}, k_{y1}, and k_{y2} in the figure. It is assumed that eight DOFs (marked with arrows) are instrumented with sensors. The elastic moduli and the spring stiffnesses require updating. Modal properties of the structure with actual/correct stiffness values, shown in Table I, are used as "experimental" properties. Only the first mode is used for model updating in this example ($n_{\text{modes}} = 1$). The bounds on the optimization variables are set as $\alpha_j \in [-0.3, 0.3]$, $\lambda_i \in \left[0.8\lambda_i^{\text{EXP}}, 1.2\lambda_i^{\text{EXP}}\right]$, and $\boldsymbol{\psi}_i \in [-2 \cdot \mathbf{1}, 2 \cdot \mathbf{1}]$. Weights are set as $w_{\lambda_i} = w_{\boldsymbol{\psi}_i} = w_{\text{eig}} = 1$. The convergence tolerance is $\xi = 10^{-5}$. On a computer with an Intel i7-9700 CPU and 16 GB of RAM, it takes approximately 0.35 seconds to find the globally optimal solution with a preprocessing local search. In Table I, the updating variables at the solution are shown as α_j^* and the actual values of the updating variables are shown as α_j^{act}. The two groups of values are nearly identical within five decimal places.

Four-Story Shear-Frame Structure with Experimental Data

Next, the algorithm is applied to experimental data from the laboratory four-story shear-frame structure shown in Figure 3. The first three floors are instrumented with accelerometers ($n_{\mathcal{M}} = 3$) and the first two modes are available for model updating ($n_{\text{modes}} = 2$). Shown in Figure 4, the four inter-story stiffnesses, $k_1, \ldots, k_4$, are updated, corresponding to the four updating variables, $\alpha_1, \ldots, \alpha_4$, which quantify the

relative changes from nominal values. The bounds on the optimization variables are set as $\alpha_j \in [-0.6,0.6]$, $\lambda_i \in \left[0.8\lambda_i^{\mathrm{EXP}}, 1.2\lambda_i^{\mathrm{EXP}}\right]$, and $\boldsymbol{\psi}_i \in [-1.7 \cdot \mathbf{1}, 1.7 \cdot \mathbf{1}]$. Weights are set as $w_{\lambda_i} = w_{\boldsymbol{\psi}_i} = 1$ and $w_{\mathrm{eig}} = 10^{-3}$. The convergence tolerance is $\xi = 10^{-5}$. On the same computer used in the previous section, it takes approximately 37 seconds to converge to a global optimum (Figure 5). In Table II, the modal properties of the nominal model and two different updated models are compared with the modes extracted from the experimental data. One model is updated based on Eq. (2) using a local search algorithm with 100 randomly generated starting points [1] while the other model is updated based on Eq. (3) using the P-RD global optimization algorithm. The analytical resonance frequency from the model is denotated as f_i and the error is calculated as:

$$\Delta f_i = \frac{\left| f_i^{\mathrm{EXP}} - f_i \right|}{f_i^{\mathrm{EXP}}} \times 100\% \tag{5}$$

where f_i^{EXP} is the experimental resonance frequency. The modal assurance criterion (MAC) is defined as:

$$\mathrm{MAC}_i = \frac{((\boldsymbol{\psi}_i^{\mathrm{EXP},\mathcal{M}})^{\mathrm{T}} \boldsymbol{\psi}_i^{\mathcal{M}})^2}{\left\| \boldsymbol{\psi}_i^{\mathrm{EXP},\mathcal{M}} \right\|_2^2 \left\| \boldsymbol{\psi}_i^{\mathcal{M}} \right\|_2^2}, \quad i = 1, \ldots, n_{\mathrm{modes}} \tag{6}$$

The MAC ranges from 0 to 1, with values closer to 1 representing vectors that are more similar to each other. It can be seen that the modal properties of both updated models match the experimental results much more closely than the nominal model.

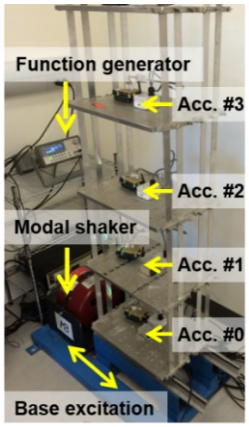


Figure 3. Test set-up.

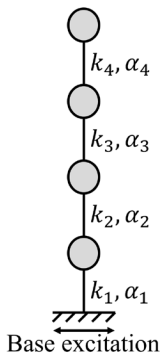


Figure 4. Model.

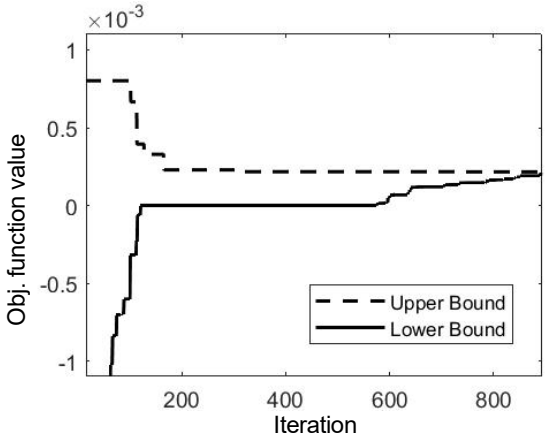


Figure 5. Convergence plot.

TABLE II. MODAL PROPERTIES OF NOMINAL AND UPDATED MODELS

Modes	Exp.	Nominal model			Local update [1]			P-RD update		
	f_i^{EXP} (Hz)	f_i (Hz)	Δf_i (%)	MAC	f_i (Hz)	Δf_i (%)	MAC	f_i (Hz)	Δf_i (%)	MAC
1st	0.88	0.99	12.0	1.0000	0.88	0.71	0.9996	0.88	0.28	0.9994
2nd	2.75	2.85	3.64	0.9317	2.77	0.68	0.9999	2.75	0.16	0.9998

CONCLUSIONS

FE model updating algorithms previously studied are unable to guarantee global optimality. Therefore, this paper investigates the application of the P-RD global optimization algorithm on the eigenvector difference formulation. The eigenvector difference problem is first reformulated into a biconvex form. The P-RD algorithm can then find a global optimum by iteratively solving the primal problem followed by the relaxed dual problem. The algorithm terminates when the upper bound of the optimal objective function value is within a specified tolerance of the lower bound or provides a certificate of the duality gap in case the computation does not converge due to limitation in computing resources. The FE model updating algorithm is validated through two examples: (i) a numerical study of a plane truss structure and (ii) experimental data of a four-story shear-frame structure. The stiffness parameters of the plane truss structure were successfully identified to a high degree of accuracy. Using experimental data of a four-story shear-frame structure, the four inter-story stiffnesses were able to be updated. The modes simulated from the updated model are shown to match the experimental modal properties much more closely than the nominal model. These examples exhibit how the updated model can better represent the behavior of the as-built structure.

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